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An integral treatment for combined heat and mass transfer by natural convection along a horizontal surface in a porous medium

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ABSTRACT

The heat and mass transfer characteristics of natural convection about a horizontal surface embedded in a saturated porous medium is analyzed. An integral procedure is derived to the heated horizontal surface, where surface temperature and surface concentration are power function of distance from the leading edge of porous plate. Local Nusselt number and local Sherwood number variations in the boundary layer are presented graphically and in the tables for the various values of problem parameters and it is found that the temperature and concentration fields near the plate increases with power law exponent n.

1. Introduction

Transport processes through porous media play important roles in diverse applications, such as in geothermal applications, petroleum industries, thermal insulation, design of solid-matrix heat exchangers, chemical catalytic reactors and many others. The study of convective heat and mass transfer and fluid flow in porous media has received great attention in recent years. The state of art concerning combined heat and mass transfer in porous media has been summarized in the excellent monographs by Nield and Bejan [1] and Ingham and Pop [2,3].

From a fundamental perspective, Nield [4] made the first attempt to study the stability of convective flow in horizontal layers with imposed vertical temperature and concentration gradients. This was followed by Khan and Zebib [5] in the study of flow stability in a vertical porous layer. Bejan and his co-workers [6–8] conducted a series of investigation of these effects in natural convection in a fluid-saturated porous medium. Other geometries considered in the previous studies include line sources [9], vertical surfaces [10–13], horizontal surfaces [14–17], vertical cylinders [18] and the slender bodies of revolution [19]. More recently, Sakamoto and Kulacki [20] reported experimental study of measurements of heat transfer coefficients in steady natural convection in a saturated porous medium. Experimental results show that heat transfer coefficient can be adequately determined via a Darcy-based model, the result confirm a correlation proposed by Bejan [21]. The Darcy model works well in a porous medium has a lower effective Prandtl number near the wall than in the bulk medium. The factors that contribute to this effect include the thinning of the boundary layer near the wall and an increase of effective thermal conductivity.

Relative to the research activity on double-diffusive natural convection flow from vertical surface with a porous medium, the work reported on horizontal natural convection driven by combined buoyancy is limited, probably owing to the mathematical complexities involved in the problem. So, this work presents the analytic solution to the coupled non-linear equations of heat and mass transfer by natural convection from the horizontal surface in a porous medium. This problem has important applications to convective flow above heated bedrock or below the cooled caprock in a liquid-dominated geothermal reservoir. An integral procedure is derived to solve the problem along the lines of Nakayama and Hossain [22]. The comparison of the present integral solutions with numerical solutions of pure thermal buoyancy-driven flow [23,24], reveals excellent performance of the approximate method.

2. Mathematical analysis

Consider the natural convection in a porous medium saturated with a Newtonian fluid on a horizontal plate (see Fig. 1). The *x*-coordinate is measured along the surface and the *y*-coordinate normal to it. It is assumed that the wall temperature and concentration are power function of distance along the plate from its leading edge, i.e. $T_w = T_\infty + ax^n$ and $C_w = C_\infty + bx^n$, where *a*, b and *n* are positive constants.

Several assumptions are used throughout the present paper: (a) the flow is steady and incompressible; (b) the physical properties are considered to be constant, except for the density term that is associated with a body force; (c) flow is sufficiently slow that the convecting fluid and the porous matrix are in local thermodynamic equilibrium; (d) the processes occur at low concentration difference such that the diffusion-thermo and thermo-diffusion effects and the interfacial velocity due to mass diffusion can be neglected; and (f) the Boussinesq approximation is valid and the boundary layer approximation is applicable.

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Nomenclature

a	constant defined by Eq. (5)
b	constant defined by Eq. (5)
C_p	specific heat, J/(kg k)
D	mass diffusivity of porous medium (m^2/s)
g	gravitational acceleration (m/s^2)
h	average heat transfer coefficient (W/m^2 K)
K	permeability (m^2)
k	thermal conductivity of the porous medium, (W/m K)
Le	Lewis number
m	average mass transfer coefficient (m/s)
Nu	local Nusselt number
N	buoyancy ratio
n	constant defined by Eq. (5)
q	local heat transfer coefficient
Ra	modified Rayleigh number
Sh	local Sherwood number
T	non-dimensional temperature
u	velocity component in the x-direction (m/s)
v	velocity component in the y-direction (m/s)
x	horizontal co-ordinate
y	vertical co-ordinate

Greek symbols

α	thermal diffusivity of porous medium (m^2/s)
β_l	coefficient of thermal expansion (l/K)
β_c	coefficient of concentration expansion (m^3/kg)
φ_p	porosity
η	Independent similarity variable
θ	dimensionless temperature
v	kinematics viscosity (m^2/s)
μ	dynamic viscosity (kg/ms)
ϕ	dimensionless concentration
ρ	fluid density, (kg m^3)
ψ	dimensional stream function
ξ	boundary layer ratio thickness

Subscripts

w	condition at the wall
∞	condition at the infinity

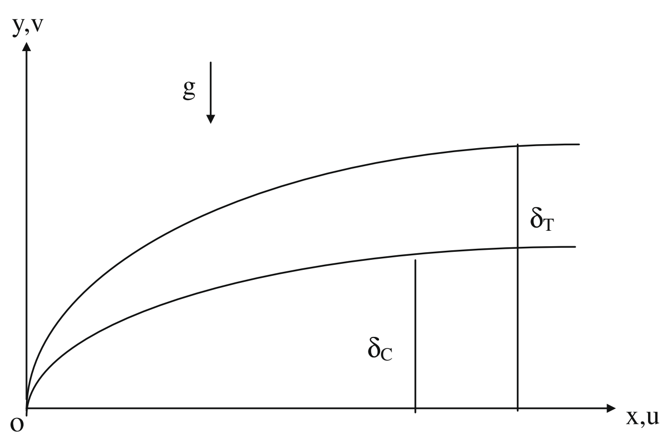

Fig. 1. Physical model and co-ordinate system.

In-line with these assumptions, the governing equations describing the conservation of mass, momentum, energy and concentration can be written as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial y} = -\frac{gK}{\gamma}\left(\beta_T \frac{\partial T}{\partial x} + \beta_C \frac{\partial C}{\partial x}\right) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} \tag{4}$$

In the above, (u, v) are the velocities in the (x, y) directions. β_T and β_C are the thermal and concentration expansion coefficient respectively, γ is the kinematic viscosity of the fluid, μ is the viscosity of the fluid, g is the gravitational acceleration, K is the permeability of the porous medium, α and D are the equivalent thermal and mass diffusivities of the porous medium, T and C are the temperature and concentration.

The boundary conditions at the wall and at infinity are, respectively

$$y = 0, \quad v = 0, \quad T_w = T_\infty + ax^n, \quad C_w = C_\infty + bx^n \tag{5}$$

$$y \to \infty, \quad u = 0, \quad T \to T_\infty, \quad C \to C_\infty \tag{6}$$

3. Integral solution

The energy equation (3) and concentration equation (4) can be integrated together with the continuity equation and the boundary conditions (5) and (6), to obtain

$$\frac{d}{dx}\int_0^\infty u(T - T_\infty)dy = -\alpha \left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{7}$$

$$\frac{d}{dx}\int_0^\infty u(C - C_\infty)dy = -D \left.\frac{\partial C}{\partial y}\right|_{y=0} \tag{8}$$

The infinity is boundary layer thickness for temperature and concentration. With the help of boundary conditions, we assume exponential profiles as follows:

$$\frac{T - T_\infty}{T_w - T_\infty} = \exp\left(-\frac{y}{\delta_T}\right) \tag{9}$$

$$\frac{C - C_\infty}{C_w - c_\infty} = \exp\left(-\frac{\xi y}{\delta_T}\right) \tag{10}$$

In the above, δ_T is arbitrary scale for the thermal boundary layer thickness whereas ξ is its ratio to the concentration boundary layer thickness. With the help of above profiles and using Eq. (2) with boundary conditions (5) and (6), we get the following velocity profiles:

$$u(x, y) = \frac{gK}{\gamma} a\beta_T x^{n-1} e^{-y/\delta_T}\left[\eta\delta_T + \frac{x}{\delta_T}\left(\frac{d\delta_T}{dx}\right)(y + \delta_T)\right] + \frac{gK}{\gamma} b\beta_C x^{n-1} e^{-\xi y/\delta_T}\left[\frac{\eta\delta_T}{\xi} + \frac{x}{\delta_T}\left(\frac{d\delta_T}{dx}\right)\left(y + \frac{\delta_T}{\xi}\right)\right]$$

Substituting velocity, temperature and concentration profiles into the above integral equations (7) and (8), to get the following ordinary differential equations:

$$\frac{agK\beta_T}{\gamma}\frac{d}{dx}\left[x^{2n-1}\delta_T\left[\delta_T\left(\frac{n}{2} + \frac{nN}{\xi(\xi+1)}\right) + x\left(\frac{d\delta_T}{dx}\right)\left(\frac{3}{4} + \frac{(2\xi+1)N}{\xi(\xi+1)^2}\right)\right]\right] = \frac{\alpha}{\delta_T}x^n \tag{11}$$

$$\frac{agK\beta_T}{\gamma}\frac{d}{dx}\left[x^{2n-1}\delta_T\left[\delta_T\left(\frac{n}{\xi+1} + \frac{nN}{2\xi^2}\right) + x\left(\frac{d\delta_T}{dx}\right)\left(\frac{\xi+2}{(\xi+1)^2} + \frac{3N}{4\xi^2}\right)\right]\right] = \frac{D\xi}{\delta_T}x^n \tag{12}$$

The closed form solution of Eqs. (11) and (12) can be possible only when δ_T may be written in the form as

$$\delta_T = \delta_* \left(\frac{agK\beta_T}{\gamma\alpha}\right)^{-1/3} x^{(2-n)/3} \tag{13}$$

where δ^* is constant. From the solution for δ_T, it is clear that if $n > 2$, δ_T decreases as x increases. Thus, the value of n *cannot* be greater than 2, otherwise boundary layer approximations will not be valid. Putting the expression (13) for δ_T in Eqs. (11) and (12), we get

$$\delta_*^3\left(\frac{4n+1}{3}\right)\left[\left(\frac{n}{2}+\frac{nN}{\xi(\xi+1)}\right)+\left(\frac{3}{4}+\frac{(2\xi+1)N}{\xi(\xi+1)^2}\right)\left(\frac{2-n}{3}\right)\right]=1 \tag{14}$$

$$\delta_*^3\left(\frac{4n+1}{3}\right)\left[\left(\frac{n}{\xi+1}+\frac{nN}{2\xi^2}\right)+\left(\frac{\xi+2}{(\xi+1)^2}+\frac{3N}{4\xi^2}\right)\left(\frac{2-n}{3}\right)\right]=\frac{1}{Le} \tag{15}$$

In the above,

$$N = \frac{\beta_c(C_w - C_\infty)}{\beta_T(T_w - T_\infty)} \quad \text{(Buoyancy ratio)}$$

$$Le = \frac{\alpha}{D} \quad \text{(Lewisn umber)}$$

The parameter N measures the relative importance of mass and thermal diffusion in the buoyancy-driven flow. It is clear that N is zero for thermal-driven flow, infinite for mass-driven flow, positive for aiding flow and negative for opposing flow.

Eqs. (14) and (15) can be combined to give the following algebraic equation of degree five for determining the boundary layer thickness ratio ξ as:

$$\begin{aligned}&\xi^5(n+2)+\xi^4(2n+4)+\frac{\xi^3}{3}(4nN-8nLe+16N-8Le+3n+6)\\&+\frac{\xi^2}{3}(8nN-4nLe-3nNLe-6NLe-16Le+8N)\\&-\xi(2nNLe+4NLe)-(2NLe+nNLe)=0\end{aligned} \tag{16}$$

As ξ is determined using Newton–Raphson method from the Eq. (16), δ^* can be obtained from Eqs. (14) and (15). Thus, the Nusselt and Sherwood numbers of our primary interest can be calculated easily as following.

The rate of heat and mass transfer from the wall to the medium are computed from

$$q = -k\left.\frac{\partial T}{\partial y}\right|_{y=0} = \frac{akx^n}{\delta_T} = \frac{akx^{(4n-2)/3}}{\delta_*\left(\frac{agK\beta_T}{\gamma\alpha}\right)^{-1/3}} \tag{17}$$

$$m = -D\left.\frac{\partial C}{\partial y}\right|_{y=0} = \frac{bDx^n\xi}{\delta_T} = \frac{bDx^{(4n-2)/3}\xi}{\delta_*\left(\frac{agK\beta_T}{\gamma\alpha}\right)^{-1/3}} \tag{18}$$

From the definition of local Nusselt number and local Sherwood number, rate of heat and mass transfer are calculated by

$$Nu = \frac{q}{Akx^{(4n-2)/3}\left(\frac{AgK\beta_T}{\gamma\alpha}\right)^{1/3}} = \frac{1}{\delta_*} \tag{19}$$

$$Sh = \frac{m}{BDx^{(4n-2)/3}\left(\frac{AgK\beta_T}{\gamma\alpha}\right)^{1/3}} = \frac{\xi}{\delta_*} \tag{20}$$

Here, the notations Nu and Sh are used for local Nusselt number and local Sherwood number, respectively. From the above formulae (19) and (20), it is clear that both Nu and Sh turn out to be equal in magnitude, for $\xi = 1$, $Le = 1$ and $N = 0$ and hence thermal and concentration boundary layers of equal thicknesses.

Our next interest is compute the horizontal velocity $u\ (x,y)$ nearer to the wall, which can be defined by $u(x,0)$ and is given by

$$u(x,0) = x^{(2n-1)/3}\left(\frac{agK\beta_T}{\gamma}\right)\left(\frac{agK\beta_T}{\gamma\alpha}\right)^{-1/3}\left(\frac{2+2n}{3}\right)\left(1+\frac{N}{\xi}\right)\delta_* \tag{21}$$

The velocity $u(x,y)$ must be increasing or at least constant with respect to x. This is possible if n is greater than or equal to 0.5. Thus, non-dimensional velocity near the wall is defined as

$$f'(0) = \frac{u(x.0)}{x^{(2n-1)}/3}\left(\frac{agK\beta_T}{\gamma}\right)\left(\frac{agK\beta_T}{\gamma\alpha}\right)^{-1/3} = \delta_*\left(\frac{2+2n}{3}\right)\left(1+\frac{N}{\xi}\right) \tag{22}$$

4. Results and discussion

In order to get the clear insight of the physical problem, the results for velocity, temperature and concentration near the horizontal wall are presented in Figs. 2–6. Before proceeding for a detail study, the results have been validated by comparing the results in the form Nu for parameter setting $Le = 1$, $N = 0$ and $0 \leqslant n \leqslant 2$, with earlier heat transfer results of Cheng and Chang [23] and Chang and Cheng [24], and it is found that they are in good agreement and shown in Table 1. Calculations have been made for a wide range of governing parameters and the results of the physical quantities for the selected values of the power law exponent n are presented in Table 2.

Figs. 2 and 3 give the variation of local Nusselt number and local Sherwood number for $Le = 2$ with $0.5 \leqslant n \leqslant 2$, in aiding and opposing regions. It is seen from these figures that both Nusselt and Sherwood number increase with n. It is observed that for a positive N, heat transfer increases and the increment depends strongly on Le. For $N > 0$, as Le increases, the heat transfer is seen to decrease. This is because of larger Le provides a thicker thermal boundary layer. A negative N provides the opposite effect. When N is fixed, the surface mass transfer consistently increases as Le increases. This is because of the concentration boundary layer increasingly thinner as Le increases. The case $N = 0$ indicates that the convection is due to thermal buoyancy alone.

Figs. 4 and 5 illustrate the variations of local Nusselt number and local Sherwood number versus the power law exponent n when N is fixed, for the various values of Lewis number. It is shown that both heat and mass transfer rates increase with increasing n. For aiding flow, the heat transfer decreases while mass transfer

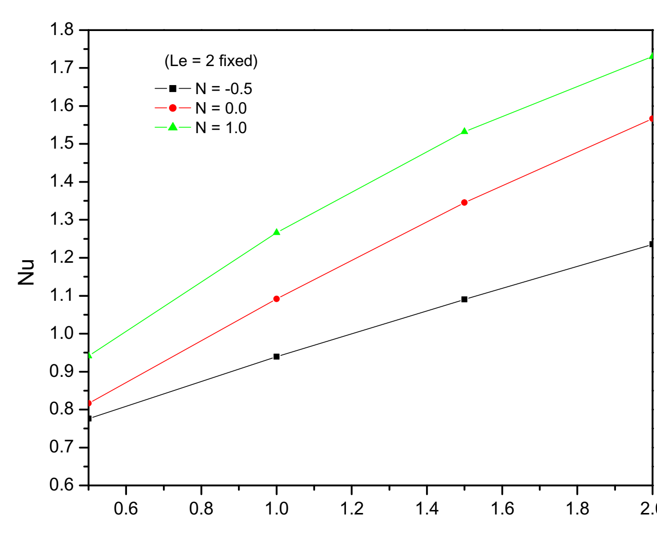

Fig. 2. Heat transfer results with n for $N = -0.5, 0, 1$.

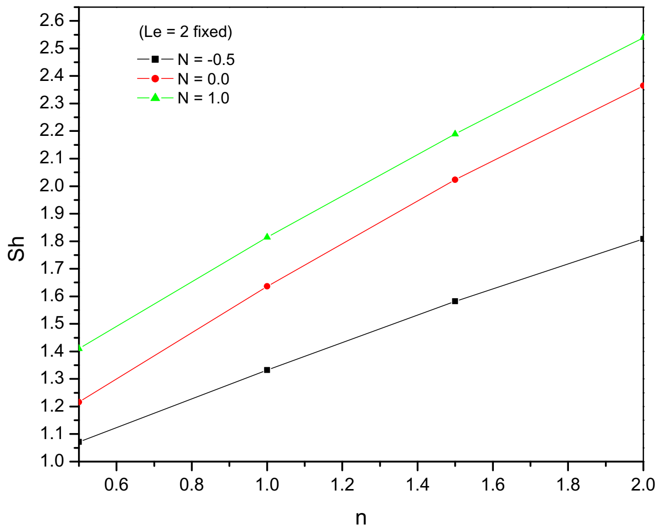

Fig. 3. Mass transfer results with n for $N = -0.5, 0, 1$.

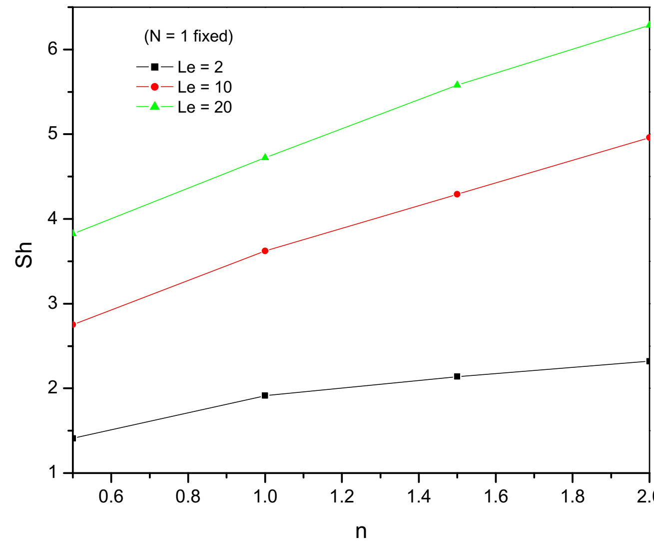

Fig. 5. Mass transfer results with n for $Le = 2, 10, 20$.

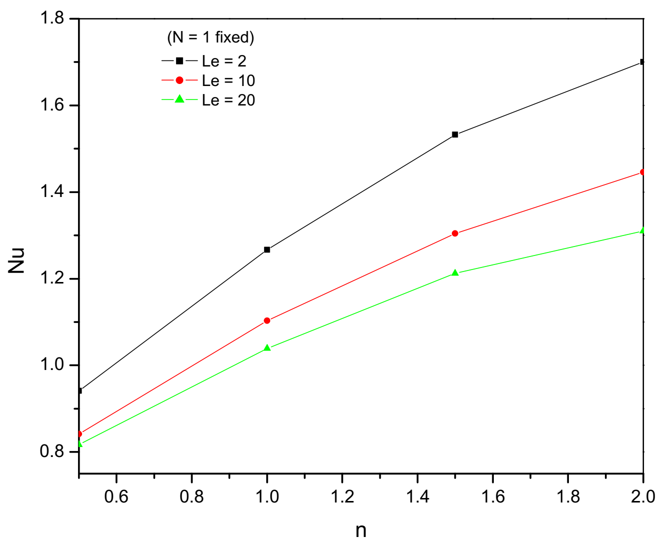

Fig. 4. Heat transfer results with n for $Le = 2, 10, 20$.

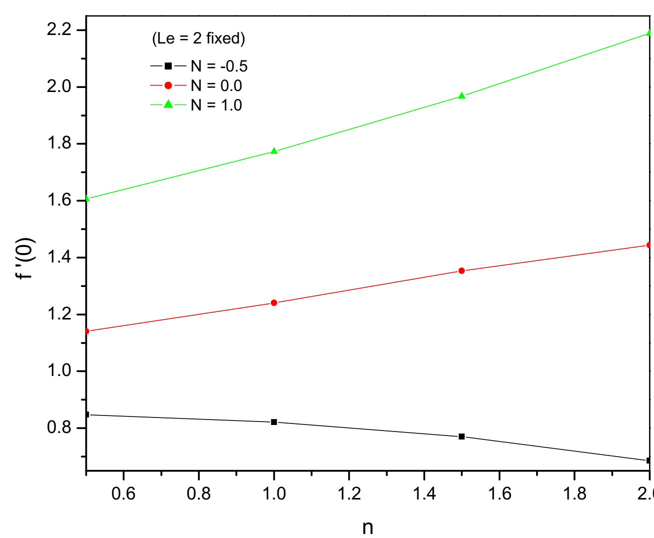

Fig. 6. The results of $f'(0)$ with n for $N = -0.5, 0, 1$.

increases significantly with increasing *Le*. Since with increasing *Le*, the concentration boundary layer adjacent to the horizontal surfaces gets thickened which suggest the possible growth in *Sh* with *Le*. For opposing flow, both heat and mass transfer rates increases with *Le*.

Figs. 6 and 7 show the effect of buoyancy ratio and Lewis number on non-dimensional velocity. It is shown that the aiding flow increases the velocity distribution while the opposing flow decreases these distributions. The velocity is found to increase with increase n for both aiding and opposing flows. It is important to note that for higher negative N, the velocity become negative and the direction of the buoyancy induced flow may reverse, which contradicts the boundary layer assumption and no solution is meaningful. For $Le > 1$, the contribution to the horizontal velocity by the mass buoyancy effect is less important and the flow reversal occurs at a smaller value of N. For $Le < 1$, the situation will be reversed for larger value of N. In the viscous fluid flow, the velocity at the wall must be zero to satisfy the no-slip condition, while in porous media, with low porosity, it is possible to obtain finite vertical velocity at the wall. This depends on the value of the wall temperature and concentration. It is important to note that the vertical velocity component should be positive for the problem under consideration and it is clear that this velocity component will become negative, which means the buoyant flow will be reversed.

5. Concluding remarks

Simultaneous heat and mass transfer by natural convection from a horizontal surface embedded in a fluid-saturated Darcian

Table 1
Values of local Nusselt number for $Le = 1$ and $N = 0$.

n	Cheng and Chang [23]	Chang and Cheng [24]	Present study
0	0.420	0.429	0.421
1/2	0.8164	0.8164	0.8164
1	1.099	1.099	1.099
3/2	1.351	1.345	1.351
2	1.571	–	1.571

Table 2
Summary of integral solution.

n	N	Le	$f'(0)$	Nu	Sh
0.5	−0.5	5	0.8134	0.7294	1.7005
		10	0.8792	0.7542	2.2944
	4	5	1.9435	1.2845	3.5240
		10	1.6603	1.1919	4.8701
1	−0.5	5	0.7793	0.9261	1.7799
		10	0.5694	0.9952	2.5360
	4	5	2.0727	1.5589	4.3795
		10	1.8161	1.4324	6.0200
1.5	−0.5	5	0.7416	1.0969	2.3917
		10	0.5262	1.2173	2.8116
	4	5	2.2435	1.7820	5.0940
		10	1.9842	1.6235	6.9614
2	−0.5	5	0.7716	1.2542	2.7382
		10	0.6591	1.3356	3.6137
	4	5	2.4080	1.9772	5.7274
		10	2.1468	1.7877	7.7815

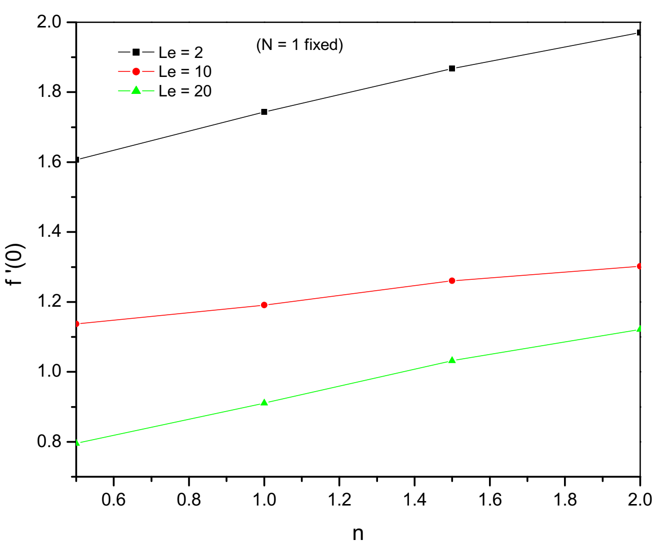

Fig. 7. The results of $f'(0)$ with n for $Le = 2, 10, 20$.

porous medium have been studied in this chapter. The original problem was solved by Von-Karman integral method. The integral solution is verified with published numerical results of heat transfer. A final comment concerning the problem of Darcy horizontal porous medium when the boundary condition at the wall $T_w = C_w$ = constant, Ref. [16] remarked that the solution presented by [14,15] for boundary layer using finite difference method are not physically plausible. The physically realistic solutions for natural convection depends on the variation of streamwise velocity, boundary layer thickness, local heat flux, local mass flux, total energy and total species convected. So, the solution presented by [14,15] is physically realistic. The task to investigate the problem for non-Darcy case of constant heat and mass boundary conditions at the wall is left for a future study.

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